Properties of Minimum Uncertainty Wavelets and Their Relations to the Harmonic Oscillator and the Coherent States

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We consider additional aspects of the recently derived "minimum uncertainty" (μ) wavelets. In particular, we show that they are fundamentally related to both the harmonic oscillator eigenstates and the canonical coherent states that play a fundamental role in quantum dynamics. In addition, we derive new raising and lowering operators that apply to the μ-wavelets. Finally, we explore in some detail the senses in which the μ-wavelets form complete sets that can be used in a variety of applications in quantum dynamics.

I. Introduction

Recently it has been shown that there exist new, relative minimum solutions of the Heisenberg uncertainty product, which we called "minimum uncertainty" (μ) wavelets.[1−6] Imposition of a constrained minimization on the Heisenberg uncertainty product leads to a hierarchical relation for generating states of decreasing uncertainty in one canonical variable from states of greater uncertainty in this variable, while producing the minimal increase in the uncertainty of the canonically conjugate variable. The role of the constraint is to prevent the variation from simply leading to a Gaussian that has been squeezed in one canonical variable. If the starting point of the hierarchy is taken to be the conventional "vacuum state" (eigenstates of the annihilation operator, $\hat{a}$, with eigenvalue zero), then one obtains the result that the μ-wavelets are a generalization of the standard vacuum (Gaussian) states. Because coherent states are eigenstates of $\hat{a}$ with, in general, complex eigenvalues α, each such state can be viewed as the vacuum state for the shifted operator[7−10]

$$\hat{A} = \hat{a} - \alpha \tag{1}$$

Consequently, for *each* member of the overcomplete set of coherent states there is a corresponding hierarchy of μ-wavelet states. Because coherent states play a fundamental role in a vast range of physics[9,11−14] (quantum field theory, quantum electrodynamics, solid state physics, statistical mechanics, etc.), as well as in mathematics, it is of considerable interest to explore more deeply the properties of μ-wavelets and their connections to coherent states. Additional impetus for such studies is provided by the fundamental role of the Heisenberg uncertainty principle in such areas as digital signal processing, filter design, etc. This paper extends the earlier analysis presented by two of the authors.[1,2]

It is useful to examine some of the reasons why the coherent states (and the harmonic oscillator eigenstates) are so widely relevant. Perelomov[8] has discussed very clearly the intimate connection between the more familiar coherent states and algebras of various quantum mechanical operators. The non-Abelian character of the operator groups underlying these algebras imposes uncertainty constraints on the precision with which the physical properties associated with the operators can be determined. Furthermore, these constraints are fundamentally responsible for the distinctive nature of quantum mechanics compared to classical dynamics. Because there is great technological potential associated with processes that follow the quantum mechanical rather than classical dynamics, one expects that it is essential to understand and be able to create and utilize maximally controlled quantum states of a wide variety of systems or materials. The greatest progress in this direction has been for spin-type systems, but there is enormous interest in similar control over matter waves. Unfortunately, there is relatively little known about *exact*, entangled solutions of the Schrödinger equation describing systems of material particles.

It is the eigenstates of physical (Hermitian) operators that provide the mathematical tools (i.e., representations and basis sets) for computations and system control, and again, the non-Abelian nature of the groups of operators representing the common observables of physical properties of material systems prevents one from having a single basis that can handle all quantities of interest. Klauder and others[7−9] have stressed the important notion that canonical coherent states provide the best possible compromise (in the minimum Heisenberg uncertainty sense) between say the coordinate and momentum representations. The "canonical" label simply stresses the fact that these are coherent states associated with noncommuting canonically

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conjugate variables. In this regard, we stress the fact that some of the most useful coherent states are those based on the Gaussian function, because these control the uncertainty in position and momentum. In fact, we shall focus solely on these in the present work, although it is an interesting question whether minimum uncertainty wavelets can also be generated for non-Gaussian coherent states associated with other types of observables.[7−9]

The plan of this paper is as follows. In the next section we give a brief summary of the μ-wavelets, the hierarchy they satisfy, and their connection to previously introduced "Hermite distributed approximating functionals" (HDAFs). Also in this section we examine new raising and lowering operators for the μ-wavelets that differ from the usual ones associated with canonical coherent states and the eigenstates of the harmonic oscillator.[3−6]

Then in section III, we examine some detailed relations between μ-wavelets and harmonic oscillator states. We shall see that μ-wavelets are eigenvectors of a non-Hermitian version of the harmonic oscillator, which is due to a similarity transformation of the harmonic oscillator Hamiltonian, where the transformation does not possess a bounded inverse.[15] This will result in our obtaining various resolutions of the identity in terms of μ wavelets. Included are resolutions in the $\mathscr{L}^2$ (Hilbert space) sense, the weak sense, and the $\mathscr{L}^2$-sense for the dense Schwarz subspace of Hilbert space. Our discussion focuses on the canonical coherent states and we follow the review of Perelomov.[8]

In section IV we discuss the relation between HDAFs[16−19] and the canonical coherent states. Then in section V we indicate some possible avenues of further inquiry and give our conclusions. Finally, in the Appendix we discuss the shift operator and establish completeness in the weak sense for *any* $\mathscr{L}^2$-function (the so-called "fiduciary function" discussed in Klauder and Skagerstam).[9]

II. Relative Minimum Uncertainty Hierarchy Defining μ-Wavelets

We shall couch our discussion in terms of the observables corresponding to the (Cartesian) position operator, $\hat{x}$, and its canonically conjugate momentum operator, $\hat{k}$, satisfying the commutation relation

$$[\hat{x}, \hat{k}] = \mathrm{i}\hat{1} \tag{2}$$

Thus, $\{\hat{x}, \hat{k}, \hat{1}\}$ are elements of a Heisenberg Lie-algebra.[7,8]

Consider the set of all ket vectors $|\phi^\sigma(0)\rangle$ centered in the phase space at $x = 0$, $k = 0$ in the sense that

$$0 = \frac{\langle\phi^\sigma(0)|\hat{x}|\phi^\sigma(0)\rangle}{\langle\phi^\sigma(0)|\phi^\sigma(0)\rangle} \tag{3}$$

and

$$0 = \frac{\langle\phi^\sigma(0)|\hat{k}|\phi^\sigma(0)\rangle}{\langle\phi^\sigma(0)|\phi^\sigma(0)\rangle} \tag{4}$$

In one dimension, the Heisenberg uncertainty principle takes the form

$$\Delta x\Delta k = \frac{\langle\phi^\sigma(0)|\hat{x}^2|\phi^\sigma(0)\rangle\langle\phi^\sigma(0)|\hat{k}^2|\phi^\sigma(0)\rangle}{\langle\phi^\sigma(0)|\phi^\sigma(0)\rangle^2} \geq \frac{1}{2} \tag{5}$$

The equality holds for the state $|\phi_0^\sigma(0)\rangle$, which satisfies the condition

$$\hat{x}|\phi_0^\sigma(0)\rangle = -\mathrm{i}\sigma^2\hat{k}|\phi_0^\sigma(0)\rangle \tag{6}$$

where σ^2 is real and greater than 0. (A family of relative minimum uncertainty solutions arise if σ^2 is complex with Re $\sigma^2 > 0$; one can always introduce new canonical operators for which such states give the absolute minimum uncertainty product.[7,8])

As is to be expected, these equations are essentially unchanged for ket vectors centered at an arbitrary point x, k in the phase space. This is conveniently demonstrated by introducing the shift operator $\hat{D}(\alpha)$.[5,7−10] For completeness, basic features of this operator are discussed in the Appendix. The action of $\hat{D}(\alpha)$, expressed in either the x or k representation, is to shift the origin and adjust the phase of the ket vector on which it acts. That is, for any Hilbert-space vector $|f\rangle$,

$$\langle x'|f(\alpha)\rangle = \langle x'|\,\hat{D}(\alpha)|f\rangle = \mathrm{e}^{-(1/2)\mathrm{i}xk}\mathrm{e}^{\mathrm{i}kx'}f(x'-x) \tag{7}$$

and

$$\langle k'|f(\alpha)\rangle = \langle k'|\hat{D}(\alpha)|f\rangle = \mathrm{e}^{(1/2)\mathrm{i}xk}\mathrm{e}^{-\mathrm{i}xk'}\tilde{f}(k'-k) \tag{8}$$

where

$$\alpha = \frac{1}{\sqrt{2}}\left[\frac{x}{\sigma} + \mathrm{i}k\sigma\right] \tag{9}$$

Here α is a complex-number representation of the phase point x, k. The quantity σ is a scaling parameter with the dimensions of length. Defined in this way

$$\hat{D}(\alpha)^{-1} = \hat{D}(\alpha)^\dagger = \hat{D}(-\alpha) \tag{10}$$

and thus the shift operator is unitary. By appropriate insertion of the identity in the form

$$\hat{1} = \hat{D}(\alpha)^\dagger\,\hat{D}(\alpha) \tag{11}$$

into this series of equations, we can transform them so as to reference them to any arbitrary x, k phase-space point. That is

$$0 = \frac{\langle\phi^\sigma(\alpha)|(\hat{x} - x)|\phi^\sigma(\alpha)\rangle}{\langle\phi^\sigma(\alpha)|\phi^\sigma(\alpha)\rangle} \tag{12}$$

$$0 = \frac{\langle\phi^\sigma(\alpha)|(\hat{k} - k)|\phi^\sigma(\alpha)\rangle}{\langle\phi^\sigma(\alpha)|\phi^\sigma(\alpha)\rangle} \tag{13}$$

$$\Delta x\Delta k = \frac{\langle\phi^\sigma(\alpha)|(\hat{x} - x)^2|\phi^\sigma(\alpha)\rangle\langle\phi^\sigma(\alpha)|(\hat{k} - k)^2|\phi^\sigma(\alpha)\rangle}{\langle\phi^\sigma(\alpha)|\phi^\sigma(\alpha)\rangle^2} \geq \frac{1}{2} \tag{14}$$

and

$$(\hat{x} - x)|\phi_0^\sigma(\alpha)\rangle = -\mathrm{i}\sigma^2(\hat{k} - k)|\phi_0^\sigma(\alpha)\rangle \tag{15}$$

In these equations, we have made use of the similarity transforms

$$\hat{D}(\alpha)\hat{x}\hat{D}(\alpha)^\dagger = \hat{x} - x$$

and

$$\hat{D}(\alpha)\hat{k}\hat{D}(\alpha)^\dagger = \hat{k} - k \tag{16}$$

Equation 15 can be written in the well-known eigenvalue form[5,7−10]

$$\hat{a}_{0,\sigma}|\phi_0^\sigma(\alpha)\rangle = \alpha|\phi_0^\sigma(\alpha)\rangle \quad (17)$$

where

$$\hat{a}_{0,\sigma} = \frac{1}{\sqrt{2}}\left[\frac{\hat{x}}{\sigma} + i\hat{k}\sigma\right] \quad (18)$$

clearly is the annihilation operator for the state $|\phi_0^\sigma(0)\rangle$. From the foregoing it is clear that $|\phi_0^\sigma(0)\rangle$ is *proportional* to $|n = 0, \sigma\rangle$, the harmonic oscillator ground state. Both are, of course, Gaussian in either the x or k representation. We distinguish between these two ket vectors because it is convenient (as later will be made clear) to apply different normalization conditions to each, namely $\langle k = 0|\phi_0^\sigma(0)\rangle = 1$ and $\langle n = 0, \sigma|n = 0, \sigma\rangle = 1$.

For our present purposes, a useful way to constrain the minimization of eq 5 (and, more generally, eq 14), which we now summarize, was first given by Hoffman and Kouri.[1,2] For convenience we consider a state centered at $x = 0$, $k = 0$, but as shown above, the state can be arbitrarily centered in the phase space using the shift operator. To simplify notation, we will assume that all ket vectors are centered at $x = 0$ and $k = 0$ unless explicitly indicated otherwise. Thus, for example, we write $|\phi_0^\sigma(0)\rangle$ simply as $|\phi_0^\sigma\rangle$. We begin with the state of minimum uncertainty, $|\phi_0^\sigma\rangle$, and add to it $|\phi_1^\sigma\rangle$, to the end that Δx for the resulting state, $|\psi_1^\sigma\rangle$, defined by

$$|\psi_1^\sigma\rangle = |\phi_1^\sigma\rangle + |\phi_0^\sigma\rangle \quad (19)$$

is decreased. We note that the starting vector can actually be arbitrary, as we will discuss shortly, but we first focus on the case where it is taken to be an absolute minimum uncertainty state. Of course, if the new state $|\psi_1^\sigma\rangle$ is not Gaussian, then the overall uncertainty product $\Delta x\Delta k$ must increase. Our object is to have this happen in a controlled way. (Note: Here we are squeezing Δx at the expense of Δk; the roles can of course be reversed.) Equation 5 can be expressed in the form

$$(\Delta x\Delta k)^2 = \Delta_f^{\ 2} + \Delta_v^{\ 2} \quad (20)$$

where we constrain the variation to all possible $|\phi_1^\sigma\rangle$ that keep Δ_f^2, defined by

$$\Delta_f^{\ 2} \equiv \frac{\langle\psi_1^\sigma|\,\hat{x}^2|\psi_1^\sigma\rangle}{\langle\psi_1^\sigma|\psi_1^\sigma\rangle^2}[\langle\phi_0^\sigma|\hat{k}^2|\phi_0^\sigma\rangle + \langle\phi_1{}^\sigma|\hat{k}^2|\phi_0^\sigma\rangle + \langle\phi_0^\sigma|\hat{k}^2|\phi_1{}^\sigma\rangle] \quad (21)$$

fixed. Subject to this condition, we vary $|\phi_1^\sigma\rangle$ so that that the *nonnegative* quantity Δ_v^2, defined by

$$\Delta_v^{\ 2} \equiv \frac{\langle\psi_1^\sigma|\,\hat{x}^2|\psi_1^\sigma\rangle\langle\phi_1^\sigma|\hat{k}^2|\phi_1^\sigma\rangle}{\langle\psi_1^\sigma|\psi_1^\sigma\rangle^2} \quad (22)$$

is a minimum. Clearly, $\Delta_v^2 > 0$ (if we exclude the trivial case where $|\phi_1^\sigma\rangle$ vanishes) and therefore the fixed value of Δ_f^2 sets the "floor" below which $\Delta x\Delta k$ cannot go. Such a variation is similar in spirit to the Lagrange method of undetermined multipliers in that Δ_f^2 depends *both* on $|\phi_0^\sigma\rangle$ and $|\phi_1^\sigma\rangle$ and is unknown until $|\phi_1^\sigma\rangle$ has been determined. The minimization of eq 22 is carried out in exactly the same manner as the original, unconstrained variation of eq 5. The result is the condition on $|\phi_1^\sigma\rangle$ that[1,2]

$$[\hat{x} + i\sigma^2\hat{k}]|\psi_1^\sigma\rangle = i\sigma^2\hat{k}|\phi_0^\sigma\rangle \quad (23)$$

Using eq 19, we then obtain

$$[\hat{x} + i\sigma^2\hat{k}]|\phi_1{}^\sigma\rangle = i\sigma^2\hat{k}|\phi_0^\sigma\rangle \quad (24)$$

Note that the operator $[\hat{x} + i\sigma^2\hat{k}]$ that appears in these equations is proportional to the annihilation operator (see eq 18) and that eq 24 is somewhat reminiscent of a lowering operation on $|\phi_1^\sigma\rangle$.

We shall now digress to consider the case where we replace $|\phi_0^\sigma\rangle$ in eq 19 with an *arbitrary* vector, $|\psi_\kappa{}^\sigma\rangle$, having the normalization $\langle k = 0|\psi_\kappa{}^\sigma\rangle = 1$, where κ is an index that denotes a particular choice of the arbitrary initial state. For notational convenience we will treat κ as though it were a numerical index and reserve $\kappa = 0$ (and, as a consequence, all integer values of κ) for the case $|\psi_0{}^\sigma\rangle \equiv |\phi_0^\sigma\rangle$, which we have just discussed. The minimization process proceeds exactly as before and in place of eq 23 we obtain

$$[\hat{x} + i\sigma^2\hat{k}]|\psi_{\kappa+1}^\sigma\rangle = i\sigma^2\hat{k}|\psi_\kappa^\sigma\rangle \quad (25)$$

If we express this equation in the k representation, where $\langle k|\hat{x}|k'\rangle = i\,\partial/\partial k\;\delta(k-k')$, and introduce the variable[2]

$$\xi \equiv \frac{k^2\sigma^2}{2} \quad (26)$$

eq 25 can be written in the form

$$e^{-\xi}\frac{\partial}{\partial\xi}[e^{\xi}\psi_{\kappa+1}(k)] = \psi_\kappa(k) \quad (27)$$

where $\psi_\kappa(k) \equiv \langle k|\psi_\kappa^\sigma\rangle$. Introducing the modified lowering operator $\hat{L}$ defined by

$$\langle k|\hat{L}|k'\rangle = e^{-\xi}\frac{\partial}{\partial\xi}[e^{\xi}\delta(k-k')] \quad (28)$$

lets us write eq 27 in the abstract form

$$\hat{L}|\psi_\kappa^\sigma\rangle = |\psi_{\kappa-1}^\sigma\rangle \quad (29)$$

This lowering operator, $\hat{L}$, is closely related to $\hat{a}$ but there exist important differences that we shall explore in the next section. The solution to the homogeneous form of eq 29 is easily shown to be $\beta|\phi_0^\sigma\rangle$, where β is a constant of integration. Using the normalization $\langle k = 0|\psi_\kappa{}^\sigma\rangle = 1$ (see ref 2), we obtain the solution

$$|\psi_{\kappa+1}^\sigma\rangle = |\phi_0^\sigma\rangle + \hat{R}|\psi_\kappa^\sigma\rangle \quad (30)$$

of eq 29 where $\hat{R}$ is a new raising operator defined by

$$\langle k|\hat{R}|k'\rangle = e^{-\xi}\int_0^{\xi} d\bar{\xi}\;[e^{-\bar{\xi}}\delta(k'-\bar{k})] \quad (31)$$

It is easily shown that the operators $\hat{L}$ and $\hat{R}$ are not exact inverses (analogous to the situation for $\hat{a}$ and $\hat{a}^\dagger$) because $\hat{L}$ has a zero eigenvalue. Instead, they obey the commutation relation

$$[\hat{L}, \hat{R}] = |\phi_0^\sigma\rangle\langle k' = 0| \quad (32)$$

The above process can be repeated using $|\psi^{\sigma}_{\kappa+1}\rangle$ as a starting state and so on. The general result is

$$\hat{L}|\psi^{\sigma}_{\kappa+1}\rangle = |\psi^{\sigma}_{\kappa}\rangle \tag{33}$$

and

$$|\psi^{\sigma}_{\kappa+n}\rangle = |\phi^{\sigma}_{0}\rangle + \hat{R}|\psi^{\sigma}_{\kappa+n-1}\rangle = \sum_{j=0}^{n-1}\hat{R}^{j}|\phi^{\sigma}_{0}\rangle + \hat{R}^{n}|\psi^{\sigma}_{\kappa}\rangle \tag{34}$$

We now define the μ-wavelet, $|\phi^{\sigma}_{\kappa+j}\rangle$, for $n \geq 1$ as

$$|\phi^{\sigma}_{\kappa+j}\rangle \equiv |\psi^{\sigma}_{\kappa+j}\rangle - |\psi^{\sigma}_{\kappa+j-1}\rangle \tag{35}$$

which from eqs 29 and 30 leads to

$$\hat{L}|\phi^{\sigma}_{\kappa+j+1}\rangle = |\phi^{\sigma}_{\kappa+j}\rangle \tag{36}$$

and

$$\hat{R}|\phi^{\sigma}_{\kappa+j}\rangle = |\phi^{\sigma}_{\kappa+j+1}\rangle \tag{37}$$

for $j \geq 1$. Starting with $|\phi^{\sigma}_{\kappa+1}\rangle$ and eq 37, we can generate an infinite family of μ-wavelets for a given κ by repeated application of the raising operation, where each application of the raising operator $\hat{R}$ results in a unit increase of the μ-wavelet index. Lower-indexed μ-wavelets can be obtained from those of higher index by application of the lowering operator. As outlined above, $|\phi^{\sigma}_{\kappa+1}\rangle$ is the family member of *lowest* index that enters naturally into the theory (although, of course, the family could be artificially extended by repeated application of the lowering operator on $|\phi^{\sigma}_{\kappa+1}\rangle$). However, for the case $\kappa = 0$, eq 24 is equivalent to

$$\hat{L}|\phi_{1}{}^{\sigma}\rangle = |\phi^{\sigma}_{0}\rangle \tag{38}$$

Thus, $|\psi^{\sigma}_{0}\rangle \equiv |\phi^{\sigma}_{0}\rangle$ is in the extended family of μ-wavelets with integer index. Furthermore,

$$\hat{L}|\phi^{\sigma}_{0}\rangle = 0 \tag{39}$$

from which it is immediate that, by extension,

$$|\phi^{\sigma}_{-n}\rangle = |\psi^{\sigma}_{-n}\rangle = 0 \qquad \text{for} \qquad n = 1, 2, 3, \text{etc.} \tag{40}$$

For these reasons, the family of wavelets with integer index are special, and consequently we refer to them as canonical μ-wavelets. From eq 34 we have for the canonical μ-wavelet case that

$$|\psi^{\sigma}_{n}(0)\rangle = \sum_{j=0}^{n}|\phi^{\sigma}_{j}\rangle = \sum_{j=0}^{n}\hat{R}^{j}|\phi^{\sigma}_{0}\rangle \tag{41}$$

The explicit form for the canonical μ-wavelets in the k representation is obtained from

$$\langle k|\phi^{\sigma}_{n}\rangle = \langle k|\hat{R}^{n}|\phi^{\sigma}_{0}\rangle \qquad \text{for} \qquad n \geq 0 \tag{42}$$

The calculation is straightforward and, when combined with the results of eq 40, yields the succinct formula

$$\langle k|\phi^{\sigma}_{n}\rangle \equiv \phi_{n}(k) = \frac{\xi^{n}}{n!}e^{-\xi} \tag{43}$$

which holds for all integer values of n, positive or negative (because $1/n!$ is zero for negative integers, yielding $\phi_n = 0$, $n < 0$). Finally, analytically summing the geometric series, eq 41, leads to

$$|\psi^{\sigma}_{n}\rangle = \sum_{j=0}^{n}|\phi_{j}{}^{\sigma}\rangle = (\hat{R} - 1)^{-1}(|\phi^{\sigma}_{n+1}\rangle - |\phi^{\sigma}_{0}\rangle) \tag{44}$$

so the state vectors $|\psi^{\sigma}_{n}\rangle$ are simply finite sums of the μ-wavelets. In fact, it was previously shown that these $|\psi^{\sigma}_{n}(0)\rangle$ are exactly the HDAFs.[1,2] We shall give an explicit formula for $(\hat{R} - 1)^{-1}$ in the next section. It is evident that the canonical μ-wavelets play a central role in the refinement of $|\psi^{\sigma}_{\kappa+n}\rangle$ with increasing n, as we will later discuss in more detail. In this regard, we remark that $|\phi^{\sigma}_{0}\rangle$ has been shown to evolve by a diffusion-type process where n plays the role of a discretized "time" variable.[2] That is $|\phi^{\sigma}_{n}\rangle$ is the evolved state at "time" n. Asymptotically, for large n, this behavior approaches true diffusion. Thus, $\hat{R}$ in eq 42 can also be viewed as a kind of discretized "time" evolution or diffusion operator. Finally, we emphasize again the fact that these various relations can be referenced to any point in phase space by means of the shift operator.

This completes our summary of the μ-wavelets derivation. We turn now to a more detailed examination of some properties of μ-wavelets, in particular their close connection to harmonic oscillator states as was already implied in our foregoing discussions.

III. Relations between μ-Wavelets and Harmonic Oscillator States

To begin we note that we can define an operator that transforms the $|n,\sigma\rangle$ harmonic oscillator eigenstates directly into the $|\phi^{\sigma}_{n}\rangle$. This is most easily seen in the coordinate representation where the harmonic oscillator state is given by

$$\langle x|n,\sqrt{2}\sigma\rangle = N_{n}\exp\left(-\frac{x^{2}}{4\sigma^{2}}\right)H_{n}\left(\frac{x}{\sqrt{2}\sigma}\right) \tag{45}$$

and, taking the Fourier transform of eq 42

$$\langle x|\phi^{\sigma}_{n}\rangle = \frac{1}{\sqrt{2\pi}\sigma n!}\left(\frac{-1}{4}\right)^{n}\exp\left(-\frac{x^{2}}{2\sigma^{2}}\right)H_{2n}\left(\frac{x}{\sqrt{2}\sigma}\right) \tag{46}$$

Here $N_n = 1/\sqrt{2^{n}n!\sqrt{2\pi}\sigma}$ is the normalization constant that normalizes the harmonic oscillator eigenstates to unity. *Equation 46 can be taken as a definition of the μ-wavelets for half-integer n* (where $n!$ has the usual Γ-function interpretation). Note that the coefficient is purely imaginary for half-integer n. Under this definition eq 42 is replaced by

$$\langle k|\phi^{\sigma}_{n}\rangle \equiv \phi_{n}(k) = [\mathrm{sgn}(k)]^{2n}\frac{\xi^{n}}{n!}e^{-\xi} \qquad n = \text{integer or half-integer} \geq 0 \tag{47}$$

Defined in this manner, the μ-wavelets satisfy eqs 36 and 37. Similar to the situation for integer n, the entire family of half-integer μ-wavelets can be generated from $|\phi^{\sigma}_{1/2}\rangle$ by repeated applications of $\hat{R}$. The coordinate representation relationship between the canonical μ-wavelets and the harmonic oscillator states thus is

$$\langle x|\phi^{\sigma}_{n}\rangle = \frac{1}{\sqrt{2\pi}\sigma n!}\left(\frac{-1}{4}\right)^{n}N_{2n}{}^{-1}\exp\left(-\frac{x^{2}}{4\sigma^{2}}\right)\langle x|2n,\sqrt{2}\sigma\rangle \tag{48}$$

In abstract form we have

$$|\phi_n^\sigma\rangle = \frac{1}{\sqrt{2\pi}\sigma n!}\left(\frac{-1}{4}\right)^n N_{2n}{}^{-1}\hat{A}_{\sigma,1}|2n,\sqrt{2}\sigma\rangle = (M_n^\sigma)^{-1}\hat{A}_{\sigma,1}|2n,\sqrt{2}\sigma\rangle \quad (49)$$

where

$$M_n^\sigma \equiv (2\pi\sigma^2)^{-1/4}\left(\frac{-1}{2}\right)^n\frac{\sqrt{(2n)!}}{n!} \quad (50)$$

In eq 49 we have introduced the transformation operator $\hat{A}_{\sigma,\lambda}$ which, in the coordinate representation, is given by

$$\langle x|\hat{A}_{\sigma,\lambda}|x'\rangle = \delta(x'-x)\exp\left(-\frac{\lambda x^2}{4\sigma^2}\right) \quad (51)$$

for $0 \le \lambda \le 1$. It is useful to define a new set of ket vectors by

$$|\eta_n^{\sigma,\lambda}\rangle = (M_n^\sigma)^{-\lambda}\,\hat{A}_{\sigma,\lambda}|2n,\sqrt{2}\sigma\rangle \quad (52)$$

so that $|\eta_n^{\sigma,0}\rangle = |2n,\sqrt{2}\sigma\rangle$ and $|\eta_n^{\sigma,1}\rangle = |\phi_n^\sigma\rangle$.

The biorthogonal complement of $|\eta_n^{\sigma,\lambda}\rangle$ is

$$^b\langle\eta_n^{\sigma,\lambda}| = (M_n^\sigma)^\lambda\langle 2n,\sqrt{2}\sigma|\hat{A}_{\sigma,\lambda}{}^{-1} \quad (53)$$

where

$$\langle x|\hat{A}_{\sigma,\lambda}{}^{-1}|x'\rangle = \delta(x'-x)\exp\left(\frac{\lambda x^2}{4\sigma^2}\right) \quad (54)$$

The operator $\hat{A}_{\sigma,\lambda}$ is nonsingular and so $\hat{A}_{\sigma,\lambda}^{-1}$ is well defined; however, it is unbounded and so its matrix elements do not converge in all representations (although they do converge in both the coordinate representation and the harmonic oscillator representation of eq 45). The ket vector $|\eta_n^{\sigma,\lambda}\rangle^b$ is in the Hilbert space (i.e., normalizable in the sense $\mathscr{L}^2$) for $0 \le \lambda < 1$. The limit $\lim_{\lambda\to 1}{}^b\langle\eta_n^{\sigma,\lambda}|$ is not a normalizable Hilbert-space vector, but the function defined by $\lim_{\lambda\to 1}{}^b\langle\eta_n^{\sigma,\lambda}|x\rangle$ exists and is simply proportional to $H_{2n}(x/\sqrt{2}\sigma)$. The biorthogonality relation

$$^b\langle\eta_m^{\sigma,\lambda}|\eta_n^{\sigma,\lambda}\rangle = \delta_{m,n} \quad (55)$$

for all λ in the range $0 \le \lambda < 1$ (and hence in the limit as $\lambda \to 1$) follows immediately from the defining equations and the orthogonality of the harmonic oscillator states of eq 45. Obviously, only for $\lambda = 0$ is the dual of $|\eta_n^{\sigma,\lambda}\rangle$ its own biorthogonal complement. One can also resolve the identity in terms of the $|\eta_n^{\sigma,\lambda}\rangle$ states, but we defer a discussion of this point until later.

It is clear that the algebraic structure of the harmonic oscillator is maintained in the $|\eta_n^{\sigma,\lambda}\rangle$ biorthogonal basis because they are connected by similarity transformations. Thus, the transformed annihilation operator, $\hat{a}_{\sigma,\lambda}$, and creation operator, $\hat{a}_{\sigma,\lambda}^\ddagger$ are given by

$$\hat{a}_{\sigma,\lambda} = \hat{A}_{\sigma,\lambda}\hat{a}_{0,\sqrt{2}\sigma}\hat{A}_{\sigma,\lambda}{}^{-1} = \frac{(1+\lambda)}{2\sigma}\,\hat{x} + \mathrm{i}\sigma\hat{k} \quad (56)$$

and

$$\hat{a}_{\sigma,\lambda}^\ddagger = \hat{A}_{\sigma,\lambda}\hat{a}_{0,\sqrt{2}\sigma}^\dagger\hat{A}_{\sigma,\lambda}{}^{-1} = \frac{(1-\lambda)}{2\sigma}\,\hat{x} - \mathrm{i}\sigma\hat{k} \quad (57)$$

Note that $\hat{a}_{\sigma,\lambda}^\ddagger$ is not the adjoint of $\hat{a}_{\sigma,\lambda}$ except for the case $\lambda = 0$. The transformed creation and annihilation operators, of course, must still obey the commutation relation

$$[\hat{a}_{\sigma,\lambda}, \hat{a}_{\sigma,\lambda}^\ddagger] = \hat{1} \quad (58)$$

The raising and lowering equations become

$$\hat{a}_{\sigma,\lambda}^\ddagger|\eta_n^{\sigma,\lambda}\rangle = \left(\frac{M_{n+1/2}^\sigma}{M_n^\sigma}\right)^\lambda\sqrt{2n+1}|\eta_{n+1/2}^{\sigma,\lambda}\rangle \quad (59)$$

and

$$\hat{a}_{\sigma,\lambda}|\eta_n^\sigma(0)\rangle = \left(\frac{M_{n-1/2}^\sigma}{M_n^\sigma}\right)^\lambda\sqrt{2n}|\eta_{n-1/2}^\sigma\rangle \quad (60)$$

In the momentum representation, these equations are consistent with the general form of $\langle k|\phi_n^\sigma\rangle$ of eq 47 when $\lambda = 1$. Note, from the above two equations, that although $\hat{a}_{\sigma,1}$ and $\hat{a}_{\sigma,\lambda}^\ddagger$ are *related* to $\hat{L}$ and $\hat{R}$, respectively, (e.g., $\langle k|\hat{a}|k'\rangle = \mathrm{i}\sigma k/\sqrt{2}\,\langle k|\hat{L}|k'\rangle$), *they are inherently different in that the former lower and raise in increments of* $^1/_2$ *whereas the latter lower and raise in unit increments*.

The transformed Hamiltonian is

$$\hat{H}_{\sigma,\lambda} = \hat{a}_{\sigma,\lambda}^\ddagger\hat{a}_{\sigma,\lambda} + \frac{1}{2}\hat{1} \quad (61)$$

and, although it is not self-adjoint in the biorthogonal representation, its eigenvalue structure is unchanged. That is, one still has

$$\hat{H}_{\sigma,\lambda}|\eta_n^{\sigma,\lambda}\rangle = \left(2n + \frac{1}{2}\right)|\eta_n^{\sigma,\lambda}\rangle \quad (62)$$

Other properties of the harmonic oscillator states also follow trivially from those of the similarity transformation.

Perhaps the most important feature of this mapping between the harmonic oscillator states and the μ-wavelets is that it establishes the conditions under which the latter are complete. Thus, *by including integer and half-integer μ-wavelets*, we can expand any Hilbert-space ket vector, $|f\rangle$, *for which the expansion coefficients* $\lim_{\lambda\to 1}{}^b\langle\eta_n^{\sigma,\lambda}|f\rangle$ *exist*. This corresponds to the dense Schwartz subspace. This is useful, for formal consideration of the properties of μ-wavelets. For example, in computing $(\hat{R} - \hat{1})^{-1}$ that appears in eq 44, we can take advantage of the fact that the $\langle k|\phi_n^\sigma\rangle$ are complete and simply consider the action of $(\hat{R} - \hat{1})$ on an arbitrary Schwartz-space basis state. From eq 27 we find that

$$\langle k|(\hat{R} - \hat{1})|\phi_n^\sigma\rangle = -\frac{\partial}{\partial\xi}\phi_{n+1}^\sigma(\xi) \quad (63)$$

Clearly, the inverse $(\hat{R} - \hat{1})^{-1}$ then is given by

$$\langle k|(\hat{R} - \hat{1})^{-1}|k'\rangle = -\int \mathrm{d}k''\,\langle k|\hat{L}|k''\rangle\int_0^{\xi''}\mathrm{d}\bar{\xi}\,\delta(\bar{k}-k') = -\int_0^{\xi}\mathrm{d}\bar{\xi}\,\delta(\bar{\xi}-\xi') - \delta(k-k') \quad (64)$$

because this operator inverts eq 63 for every basis vector $|\phi_n^\sigma\rangle$. As another example, we can use completeness to find a closed form expression for $|\psi_{\kappa+n}^\sigma\rangle$ of eq 34. Thus, starting from the expansion of an initial Schwartz-space state (assuming the γ_n coefficients exist)

$$|\psi_\kappa^\sigma\rangle = \sum_n \gamma_n |\phi_n^\sigma\rangle \tag{65}$$

we have immediately that $|\psi_{\kappa+n}^\sigma\rangle$, the state resulting from n iteration steps, is given in closed form by

$$|\psi_{\kappa+n}^\sigma\rangle = |\phi_0^\sigma\rangle + \hat{R}|\psi_{\kappa-1}^\sigma\rangle = \sum_{j=0}^{n-1} \hat{R}^j |\phi_0^\sigma\rangle + \sum_{j=0}^{\infty} \gamma_j |\phi_{\kappa+n+j}^\sigma\rangle \tag{66}$$

How the initial state evolves under repeated iterations of the constrained minimization procedure then becomes quite apparent.

The question of the practical usefulness of expansions using μ-wavelets as a basis set is, however, another matter. Formally, we have expanded the ket vector $|f\rangle$ through application of the identity in the form

$$\hat{1} = \sum_{n=0}^{\infty} |\eta_n^{\sigma,\lambda}\rangle^b \langle \eta_n^{\sigma,\lambda}| \qquad \text{for} \qquad 0 \le \lambda < 1 \tag{67}$$

As an abstract operator, for $\lambda = 1$ this expression only has meaning in the sense of the $\lambda \to 1$ limit because ${}^b\langle\eta_n^{\sigma,1}|$ cannot be normalized; however, the expansion is valid provided $\lim_{\lambda\to 1} \langle n, \sqrt{2}\sigma,\lambda|f\rangle$ exists. In essence all we have done is to expand a function $f(x) \equiv \langle x|f\rangle$ by

- first multiplying it by $\exp(\lambda x^2/4\sigma^2)$,
- then expanding the result in a harmonic oscillator basis (assuming the expansion coefficients of $\exp(\lambda x^2/4\sigma^2)\, f(x)$ exist), and
- finally multiplying that result by $\exp(-\lambda x^2/4\sigma^2)$.

Whether this is a computational efficient procedure, of course, depends strictly on $f(x)$. However, without question it expands the domain on which the function is to be represented and concomitantly tightens the range of the expansion functions about the fixed origin. In fact, we have evidence that such expansions may be especially useful, e.g., in treating certain mechanical systems such as vibrating rods and plates.[20] Another promising application appears to be for Hermite polynomial representations of the spectral density operator for filter diagonalization.[21]

Of course, we could also write the identity in the abstract form

$$\hat{1} = \sum_{n=0}^{\infty} |\eta_n^{\sigma,\lambda}\rangle^b \langle \eta_n^{\sigma,\lambda}| \qquad \text{for} \qquad 0 \le \lambda < 1 \tag{68}$$

where $|\eta_n^{\sigma,\lambda}\rangle^b$ is the dual of ${}^b\langle\eta_n^{\sigma,\lambda}|$. This amounts to just reversing the order of application of the points of procedure above, which of course simply yields an expansion of $f(x)$ in terms of Hermite polynomials. For Hilbert-space functions there are no convergence issues.

In either case, as a practical matter we are constrained to representing $f(x)$ in some region about a fixed origin, and this can be a severe limitation in, for example, scattering calculations where we need to describe a wave packet over a substantial spatial region. In the next section we explore how the basis functions can be shifted so as to avoid the constraints of a fixed origin.

IV. Distributed Approximating Functionals and Coherent States

Although, in principle, a very broad class of functions can be represented on the whole line from $-\infty$ to $+\infty$ by an expansion in Hermite polynomials, in practice such an expansion is most useful when a representation of the function in the vicinity of the origin is desired. If we want to represent an abstract ket vector about an arbitrary phase point $\bar{x}$, $\bar{k}$, we can easily do so by using the shift operator to write the identity in the form

$$\hat{1} = \hat{D}(\bar{\alpha})\hat{1}\hat{D}(-\bar{\alpha}) = \hat{D}(\bar{\alpha})\sum_{n=0}^{\infty}{}^b|\eta_n^{\sigma,\lambda}\rangle\langle\eta_n^{\sigma,\lambda}|\hat{D}(-\bar{\alpha}) = \hat{1} \qquad \text{for} \qquad 0 \le \lambda < 1 \tag{69}$$

This is an exact relation when all terms of the infinite sum are retained, but it is also of practical usefulness for evaluating $\langle x|f\rangle$ near $\bar{x}$, and similarly $\langle k|f\rangle$ near $\bar{k}$. Under these circumstances we can expect that relatively few terms in the sum are required, and thus we can approximate the identity by

$$(\hat{I}_m(\bar{\alpha},\lambda) \equiv \hat{D}(\bar{\alpha})\sum_{n=0}^{m}{}^b|\eta_n^{\sigma,\lambda}\rangle\langle\eta_n^{\sigma,\lambda}|\hat{D}(-\bar{\alpha}) \approx \hat{1} \qquad \text{for} \qquad 0 \le \lambda < 1 \tag{70}$$

The shift parameter $\bar{\alpha}$ (i.e., $\bar{x}$ and $\bar{k}$) can be chosen as is convenient for a particular purpose. The x, x' matrix element of $\hat{I}_m(\bar{\alpha},\lambda)$ in the coordinate representation is

$$\langle x|\hat{I}_m(\bar{\alpha},\lambda)|x'\rangle = \sum_{n=0}^{m}\langle x-\bar{x}|\eta_n^{\sigma,\lambda}\rangle^b e^{i\bar{k}(x-x')}\langle\eta_n^{\sigma,\lambda}|x'-\bar{x}\rangle \tag{71}$$

We obtain the HDAF approximation to the identity from this equation by taking the limit $\lim_{\lambda\to 1}$ and setting $\bar{x} = x$ and $\bar{k} = k(x)$ (because a different $\bar{k}$ can be used for each value of x) to obtain

$$\delta_m(x-x'|\sigma,\bar{k}) = \lim_{\lambda\to 1}\sum_{n=0}^{m}\langle 0|\eta_n^{\sigma,\lambda}\rangle^b e^{i\bar{k}(x-x')}\langle\eta_n^{\sigma,\lambda}|x'-x\rangle \tag{72}$$

Note that, because $\langle x|\eta_n^{\sigma,\lambda}\rangle^b$ is an odd function for half-integer n, only integer values of n enter in the sum. This leads to a coordinate-representation expression for the identity in terms of μ-wavelets. With increasing m the width of $\delta_m(x-x',\sigma)$ is squeezed with minimal spreading in the k representation in the sense of the constrained variation principle of eq 14. We stress that the functional form of k in terms of x is totally arbitrary and can be set in any way that is convenient for the problem. (See ref 22 for a discussion of a method for doing this using classical dynamics.) It is also to be stressed that $\delta_m(x-x'|\sigma,\bar{k})$ is *not* a standard coordinate representation of an approximation of the identity operator and the μ-wavelets are not being used here as a basis set in a conventional sense. In fact, every x point has its own basis.[23] An important consequence is that the HDAF approach gives a pointwise (as opposed to $\mathcal{L}^2$) approximation in the x-representation to any Hilbert-space state vector to which it is applied.

The HDAF approximation to the function then is

$$f(x) \approx \int dx'\, \delta_m(x-x'|\sigma,\bar{k})\, f(x') \tag{73}$$

In practical applications it is convenient to discretize the integral to obtain

$$f(x) \approx \sum_j \delta_m(x-x_j|\sigma,\bar{k})\, f(x_j) \tag{74}$$

which requires knowing the function to be approximated on a grid. A rigorous mathematical analysis of the HDAF approximation to Hilbert-space functions, and any number of derivatives, has been given by Chandler and Gibson.[24] In particular, they prove that the HDAF yields uniform, pointwise convergent approximations to appropriately defined classes of $\mathscr{L}^2$-functions and any number of derivatives almost everywhere. Of course, there is nothing unique about the coordinate representation. Features of the HDAF approximation, including how to carry out the discretization for accurate computation of a function and its various derivatives and how to calculate the action of various operators, have been extensively discussed in a number of papers to which the interested reader is referred.[16−19,22,23] The whole development obviously can be given where the roles of $\hat{x}$ and $\hat{k}$ are interchanged so that the spread in the k representation is squeezed at minimal expense in the x representation.

Another point to be mentioned concerns the coherent state representation of the identity, to which the HDAF bears certain similarities.[9] It is well-known that the $|\phi_0^\sigma(\alpha)\rangle$ satisfying eq 17 are an overcomplete basis in Hilbert space, with a resolution of the identity given by

$$\hat{1} = \frac{1}{\pi\langle\phi_0^\sigma(0)|\phi_0^\sigma(0)\rangle}\int_{\text{all }\alpha} \mathrm{d}\alpha\; |\phi_0^\sigma(\alpha)\rangle\langle\phi_0^\sigma(\alpha)| \tag{75}$$

The integration is two-dimensional over the real and imaginary parts of all complex eigenvalues, α (corresponding to a classical phase space integral). Because the coherent states are overcomplete one can only establish convergence in a weak sense. Nevertheless, it has been shown that the coherent states can be "sampled" discretely in the α-index and, provided the sampling is sufficiently fine, the resulting subset of coherent states remains overcomplete. This is accomplished by dividing the phase space, $(x, k) \equiv \alpha$ into regular cells of area S, such that $S \leq 2\pi\hbar$, and sampling one coherent state within each cell. In the case where $S = 2\pi\hbar$, the set remains complete if any one state is deleted, but it becomes incomplete when two or more states are removed.[8] The ket vector, $|\phi_0^\sigma(\alpha)\rangle$, in both the coordinate or momentum representation, is a Gaussian with a variance determined by σ^2 in x and σ^{-2} in k. Of course, the uniqueness of the Gaussian can be viewed as a consequence of the fact that eq 15 follows from insisting that $|\phi_0^\sigma(\alpha)\rangle$ be a state of absolute minimum uncertainty.

A comment on the term "coherent states" is in order at this point. Sometimes it is used to refer to any set of states that provides a "tight-frame" resolution of the identity such as that given by eq 75. A distinctly different usage (and the one from which the term derives) refers to states that evolve coherently, i.e., preserve their Gaussian form under free or harmonic propagation. (For example, a particle of mass m, initially in the state $|\phi_0^\sigma(\alpha)\rangle$ and freely propagating for a time, t, evolves so that it is centered at $x(t) = x + (\hbar k/m)t$ and k in the classical phase space. Furthermore, it retains its Gaussian shape in both the x and k representations although it spreads according to $\sigma^2 \rightarrow \sigma^2 + \mathrm{i}\hbar t/m$. The evolved state is no longer of absolute minimum uncertainty, but it is of relative minimum uncertainty under an appropriate constraint.) Sometimes the term coherent state is simply used to refer to the set of vectors $\{|\phi_0^\sigma(\alpha)\rangle\}$, which satisfy both of the first two conditions and additionally are of minimum uncertainty. When the concept of coherent states is generalized, the set of vectors $\{|\phi_0^\sigma(\alpha)\rangle\}$ are generally referred to as "canonical coherent states".[9] Because the μ-wavelets also derive from a (constrained) minimization and furthermore have similar propagation properties, it is an attractive idea to consider if they can be used in some way as coherent states. This can indeed be done in several ways.

It is known,[9] but apparently not widely appreciated, that the "tight-frame" resolution of the identity is not a demanding condition and in this sense a very broad set of states can be taken to be coherent states. This fact is obfuscated because the usual demonstration of eq 75 relies on the specific form of the representation of $|\phi_0^\sigma(\alpha)\rangle$ in terms of unshifted harmonic oscillator states In fact, it is easy to show that any set of ket-vectors of the form

$$\langle x|g(\bar{\alpha})\rangle = \mathrm{e}^{-\mathrm{i}\bar{k}\bar{x}/2}\mathrm{e}^{\mathrm{i}\bar{k}x}g(x-\bar{x}) \tag{76}$$

satisfy a tight-frame completeness relation. Here

$$g(x) = \langle x|g(0)\rangle \tag{77}$$

Note that the function g must be square integrable for $|g(\alpha)\rangle$ to lie in the Hilbert space. (We stress that in writing $|g(\alpha)\rangle$ as a function of the complex variable α there is *no* implication of analyticity in α in any representation of the abstract vector.) To establish the completeness relation we note that

$$\begin{aligned}\int \mathrm{d}\bar{\alpha}\; |g(\bar{\alpha})\rangle\langle g(\bar{\alpha})| &= \int \mathrm{d}x \int \mathrm{d}x' \int \mathrm{d}\bar{\alpha}\; |x\rangle\langle x|g(\bar{\alpha})\rangle\langle g(\bar{\alpha})|x'\rangle\langle x'| \\ &= \pi\int_{-\infty}^{\infty} \mathrm{d}y\; |g(y)|^2 \int \mathrm{d}x \int \mathrm{d}x'\; |x\rangle\delta(x-x')\langle x'| = \\ &\qquad \pi\langle g(0)|g(0)\rangle\hat{1}\end{aligned} \tag{78}$$

(More generally, the identity can be resolved by $\int \mathrm{d}\alpha\; |g'(\alpha)\rangle\langle g(\alpha)|$ provided only that the states $|g(\alpha)\rangle$ and $|g'(\alpha)\rangle$ are not orthogonal). It is clear that the set of coherent states $\{|g(\alpha)\rangle\}$ are not linearly independent because

$$|g(\alpha)\rangle = \frac{1}{\pi\langle g(0)|g(0)\rangle}\int \mathrm{d}\bar{\alpha}\; |g(\bar{\alpha})\rangle\langle g(\bar{\alpha})|g(\alpha)\rangle \tag{79}$$

and thus the set is overcomplete. It is noteworthy that the above implies the existence of infinitely many reproducing kernels (RPK)[9,25] and concomitant RPK Hilbert spaces (which are dense subspaces of the full $\mathscr{L}^2$-Hilbert space). To establish that the set $\{|g(\alpha)\rangle\}$ are coherent states (in the generalized sense of being able to resolve the identity[9]), nothing was required except that $g(x)$ be square-integrable. Thus, one has great flexibility in defining coherent states in terms of resolving the identity (see eq 78). The utility of canonical coherent states over other choices lies in their propagation and uncertainty properties. That is, $|\phi_0^\sigma(\alpha)\rangle \equiv \hat{D}(\alpha)|\phi_0^\sigma(0)\rangle$ is tightly centered (*and in terms of the uncertainty principle optimally centered*) around the classical phase point x and k. Whether μ-wavelet coherent states, which satisfy similar properties, can be similarly exploited to useful ends is a topic under current investigation.

V. Discussion

In this paper we have explored the connections between μ-wavelets and the harmonic oscillator eigenstates. The analysis was first done by assuming an origin, in phase space, at $\bar{x} = \bar{k} = 0$. By use of the shift operator,[9] $\hat{D}(\alpha)$, the results can be expressed relative to an arbitrary point in phase space, $\alpha = (\bar{x}, \bar{k})$. It was found that the μ-wavelets are associated with new raising ($\hat{R}$) and lowering ($\hat{L}$) operators whose algebra differs from that of the standard $\hat{a}$, $\hat{a}^\dagger$ of the harmonic oscillator. However, like the latter, they are fundamentally consequences of the minimization of the Heisenberg uncertainty principle, with the essential difference being the μ-wavelet states provide *relative* minimum Heisenberg uncertainty products. One reflection of the distinction between $(\hat{L}, \hat{R})$ and $(\hat{a}, \hat{a}^\dagger)$ is that the former maintain the symmetry of the μ-wavelets whereas the latter always change the symmetry of the harmonic oscillator state on which they act. This is a consequence of the fact that the μ-wavelet index is always $^1/_2$ that of the corresponding harmonic oscillator state, whereas $\hat{L}$ and $\hat{R}$ shift the index by one. The HDAF $|\psi_n^\sigma\rangle$ results from application of the operator $\sum_{j=0}^{n} \hat{R}^j$ to the μ-wavelet $|\phi_0^\sigma\rangle$. This geometric series is easily summed by writing it as $\sum_{j=0}^{\infty}\hat{R}^j - \sum_{j=n+1}^{\infty}\hat{R}^j$, which gives eq 44 for the "HDAF-operator".

Of considerable interest in quantum mechanics for the purposes of approximations and computations is the issue of completeness of sets of states. We have shown herein that μ-wavelets (and HDAFs) admit several ways in which they can be used to approximate states systematically and controllably. To date, the most intensely studied of these[23,24] is in the sense of so-called weak or pointwise completeness. Although this type of convergence does not preserve, e.g., properties that depend on convergence in the $\mathcal{L}^2$-norm, it has proven to be robust computationally for solving both linear and nonlinear partial differential equations.[26] Convergence in the sense of the $\mathcal{L}^2$-norm can be achieved in several ways. One is to Schmidt-orthogonalize the μ-wavelets (i.e., construct the appropriate biorthogonal sets of states by the usual method of inverting the overlap matrix of the μ-wavelets). If one restricts consideration to the dense subspace of $\mathcal{L}^2$-functions that decay faster than any polynomial (the Schwartz space), then μ-wavelets are the dual functions for an expansion in Hermite polynomials; if one expands the Schwartz space in μ-wavelets, then the Hermite polynomials play the role of the dual functions.

We also show that μ-wavelets are one of the limiting cases of a continuous *family* of biorthogonal basis functions, the other limit being complete orthonormal harmonic oscillator states. The family is indexed by the parameter λ, ranging from $\lambda = 0$ for the harmonic oscillator to $\lambda = 1$ for the μ-wavelets. Yet another resolution of the identity follows from using the μ-wavelets as coherent states. In this case, one may select *any* particular μ-wavelet and generate an overcomplete set of states by applying $\hat{D}(\alpha)$ for all possible complex values of α. Although such coherent state resolutions of the identity can be created using *any* $\mathcal{L}^2$-function, particulary useful ones (the canonical coherent states[9]) are those generated from the Gaussian. The μ-wavelets belong to this particular family of coherent states as a consequence of their being minimum Heisenberg uncertainty states. We believe that the fact that one can construct coherent state resolutions of the identity using *any* $\mathcal{L}^2$-function is not widely appreciated even though the proof is significantly simpler than the standard one given in text books of completeness of the canonical coherent states (which relies on expressing the states in the harmonic oscillator orthonormal basis). We therefore also give in the Appendix the corresponding proof for an arbitrary set of coherent states, generated from any $\mathcal{L}^2$-state, $|g(0)\rangle$. An inevitable conclusion of this analysis is that coherent states are so generic that *only* when they are generated from any $\mathcal{L}^2$-function that has some special dynamical significance will they be of particular use. The canonical coherent states, being based on the absolute minimum uncertainty Gaussian, certainly satisfy this condition. We contend that μ-wavelets, as relative minimum uncertainty states (and therefore as "generalized Gaussians") share this character.

The next step in our study of such states will focus on the multidimensional generalization of the μ-wavelets.[18] These promise to be of significant interest because the corresponding non-Cartesian DAFs (NCDAFs) are already known to be examples of multidimensional, entangled quantum states. This will be discussed elsewhere.[27]

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Appendix. Some Explicit Properties of the Shift Operator, $\hat{D}(\alpha)$

Consider an abstract state vector, $|f\rangle$ with x-component $\langle x|f\rangle = f(x)$. Assuming that $f(x)$ admits to a Taylor series expansion, we have that

$$f(x-\bar{x}) = e^{-\bar{x}\partial/\partial x} f(x) = \langle x|e^{-i\bar{x}\hat{k}}|f\rangle \tag{80}$$

The origin of $\tilde{f}(k)$ can be similarly shifted in the k-representation where $\tilde{f}(k)$ is the Fourier transform of $f(x)$. The state vector

$$e^{i\bar{k}\hat{x}}e^{-i\bar{x}\hat{k}}|f\rangle \tag{81}$$

can thus be thought of as first centering (in the above sense) the abstract vector at $\bar{x}$ and the resulting vector around $\bar{k}$. In the x-representation we have that

$$\langle x|e^{i\bar{k}\hat{x}}e^{-i\bar{x}\hat{k}}|f\rangle = e^{i\bar{k}x}f(x-\bar{x}) \tag{82}$$

and in the k-representation

$$\langle k|e^{i\bar{k}\hat{x}}e^{-i\bar{x}\hat{k}}|f\rangle = e^{-i\bar{x}(k-\bar{k})}\,\tilde{f}(k-\bar{k}) \tag{83}$$

The phase factor, $e^{i\bar{x}\bar{k}}$, appears in the k- but not the x-representation because of the order in which the shifting operations were performed. It is convenient to split this phase factor democratically between the two representations by defining a shift operator

$$\hat{D}(\alpha) \equiv e^{-(i/2)kx}e^{ik\hat{x}}e^{-ix\hat{k}} = e^{\alpha\hat{a}_{0,\sigma}{}^\dagger - \alpha^*\hat{a}_{0,\sigma}} \tag{84}$$

which produces a "shifted" ket-vector according to

$$|f(\alpha)\rangle \equiv \hat{D}(\alpha)|f\rangle \tag{85}$$

Here we follow the convention

$$\alpha \equiv \frac{1}{\sqrt{2}}\left(\frac{x}{\sigma} + i\sigma k\right) \tag{86}$$

where σ is a positive, real number that fixes the scale length for the state. (For notational convenience in eqs 84−86 we have dropped the overbar on x and k. When a label is needed, we will use the same designation on x, k, and α, for example, $\bar{x}$, $\bar{k}$, and $\bar{\alpha}$.) The operators $\hat{a}_{0,\sigma}$ and $\hat{a}_{0,\sigma}^\dagger$ are the standard lowering and raising operators

$$\hat{a}_{0,\sigma} \equiv \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\sigma} + i\sigma\hat{k}\right) \tag{87}$$

and

$$\hat{a}_{0,\sigma}^{\dagger} \equiv \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\sigma} - \mathrm{i}\sigma\hat{k}\right) \quad (88)$$

From eq 84 we have that $\hat{D}(\alpha)$ can be written in the two equivalent forms

$$\hat{D}(\alpha) = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha\hat{a}_{0,\sigma}^{\dagger}}\mathrm{e}^{-\alpha^*\hat{a}_{0,\sigma}} = \mathrm{e}^{|\alpha|^2/2}\mathrm{e}^{-\alpha^*\hat{a}_{0,\sigma}}\mathrm{e}^{\alpha\hat{a}_{0,\sigma}^{\dagger}} \quad (89)$$

from which it is immediate that

$$[\hat{D}(\alpha)]^{\dagger} = \hat{D}(-\alpha) = [\hat{D}(\alpha)]^{-1} \quad (90)$$

and hence that $\hat{D}(\alpha)$ is unitary. The zero subscript in eqs 87 and 88 is included because we will have reason to make use of shifted lowering and raising operators given by

$$\hat{a}_{\alpha,\sigma} \equiv \hat{D}(\alpha)\hat{a}_{0,\sigma}[\hat{D}(\alpha)]^{\dagger} = \hat{a}_{0,\sigma} - \alpha\hat{1} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x} - x}{\sigma} + \mathrm{i}\sigma[\hat{k} - k]\right) \quad (91)$$

and

$$\hat{a}_{\alpha,\sigma}^{\dagger} \equiv \hat{D}(\alpha)\hat{a}_{0,\sigma}^{\dagger}[\hat{D}(\alpha)]^{\dagger} = \hat{a}_{0,\sigma}^{\dagger} - \alpha\hat{1} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x} - x}{\sigma} - \mathrm{i}\sigma[\hat{k} - k]\right) \quad (92)$$

If it is also the case that the power series of $g(x)$ is everywhere convergent, then from eq 85

$$\langle x|g(\bar{\alpha})\rangle = \langle x|\hat{D}(\bar{\alpha})|g(0)\rangle \quad (93)$$

Under these circumstances, it is of some interest to resolve $|g(\alpha)\rangle$ in harmonic oscillator states $|n,\sigma\rangle$, $n = 0, 1, 2, ...$ In this representation[5,6,9,11,12]

$$|g(\alpha)\rangle = \sum_{n,n'}|n,\sigma\rangle[\hat{D}(\alpha)]_{n,n'}g_{n'}(0) \quad (94)$$

where

$$g_{n'}(0) \equiv \langle n',\sigma|g(0)\rangle \quad (95)$$

The matrix element $[\hat{D}(\alpha)]_{n,n'}$ is conveniently evaluated, using the second form in eq 89, to obtain

$$[\hat{D}(\alpha)]_{n,n'} \equiv \langle n,\sigma|\hat{D}(\alpha)|n',\sigma\rangle = \mathrm{e}^{|\alpha|^2/2}\sum_j \frac{(-\alpha^*)^j}{j!}\,\frac{\alpha^{n-n'+j}}{(n-n'+j)!}\,\frac{(n+j)!}{\sqrt{n!n'!}} \quad (96)$$

The last equality follows from expanding $\mathrm{e}^{\alpha\hat{a}_{0,\sigma}^{\dagger}}$ and $\mathrm{e}^{-\alpha^*\hat{a}_{0,\sigma}}$ in Taylor series and making use of the standard "lowering" and raising operations

$$\hat{a}_{0,\sigma}|n,\sigma\rangle = |n-1,\sigma\rangle n^{1/2} \quad (97)$$

and

$$\hat{a}_{0,\sigma}^{\dagger}|n,\sigma\rangle = |n+1,\sigma\rangle(n+1)^{1/2} \quad (98)$$

The sum in eq 96 can be performed explicitly to obtain the equivalent forms

$$[\hat{D}(\alpha)]_{n,n'} = \frac{\mathrm{e}^{|\alpha|^2/2}}{\sqrt{n!n'!}}\left\{\frac{\partial^{n'}}{\partial\alpha^{n'}}\frac{\partial^{n}}{\partial\gamma^{n}}\mathrm{e}^{\gamma\alpha}\right\}_{\gamma=-\alpha^*} = \frac{\mathrm{e}^{|\alpha|^2/2}}{\sqrt{n!n'!}}\left\{\frac{\partial^{n}}{\partial\gamma^{n}}[\gamma^{n'}\mathrm{e}^{\gamma\alpha}]\right\}_{\gamma=-\alpha^*} = \frac{\mathrm{e}^{|\alpha|^2/2}}{\sqrt{n!n'!}}\left\{\frac{\partial^{n'}}{\partial\alpha^{n'}}[\alpha^{n}e^{\gamma\alpha}]\right\}_{\gamma=-\alpha^*} \quad (99)$$

from which it can immediately be verified explicitly in this representation that $\hat{D}(-\alpha)$ is the inverse of $\hat{D}(\alpha)$. That is

$$\sum_{n'}[\hat{D}(\alpha)]_{n,n'}[\hat{D}(-\alpha)]_{n',n''} = \frac{\mathrm{e}^{|\alpha|^2}}{\sqrt{n!n''!}}\sum_{n'}\frac{1}{n'!}\left\{\frac{\partial^{n'}}{\partial\alpha^{n'}}\frac{\partial^{n}}{\partial\gamma^{n}}\frac{\partial^{n''}}{\partial\bar{\alpha}^{n''}}\frac{\partial^{n'}}{\partial\bar{\gamma}^{n'}}\mathrm{e}^{\gamma\alpha+\bar{\gamma}\bar{\alpha}}\right\}_{\gamma=-\alpha^*,\bar{\alpha}=-\alpha,\bar{\gamma}=\alpha^*} \quad (100)$$

and making use of the fact that $\mathrm{e}^{\gamma\alpha+\bar{\gamma}\bar{\alpha}}$ is an eigenfunction of $\partial/\partial\alpha\ \partial/\partial\bar{\gamma}$ with eigenvalue $\gamma\bar{\alpha}$, we find that

$$\sum_{n'}[\hat{D}(\alpha)]_{n,n'}[\hat{D}(-\alpha)]_{n',n''} = \frac{\mathrm{e}^{|\alpha|^2}}{\sqrt{n!n''!}}\left\{\frac{\partial^{n}}{\partial\gamma^{n}}[(\gamma+\bar{\gamma})^{n''}\mathrm{e}^{\gamma\bar{a}+\gamma\alpha+\bar{\gamma}\bar{a}}]\right\}_{\gamma=-\alpha^*,\bar{a}=-\alpha,\bar{\gamma}=\alpha^*} = \delta_{n,n''} \quad (101)$$

The final equality results from the fact that after differentiation no terms survive that have either $(\gamma + \bar{\gamma})$ or $(\alpha + \bar{\alpha})$ as a factor.

The last two expressions of eq 99 can be written in the form

$$[\hat{D}(\alpha)]_{n,n'} = \frac{\mathrm{e}^{y/2}}{\sqrt{n!n'!}}\mathrm{e}^{\mathrm{i}(n'-n)\varphi(-\alpha^*)}y^{(n'-n)/2}\,\frac{\mathrm{d}^n}{\mathrm{d}y^n}\,[y^{n'}\mathrm{e}^{-y}] = \frac{\mathrm{e}^{y/2}}{\sqrt{n!n'!}}\mathrm{e}^{\mathrm{i}(n-n')\varphi(\alpha)}y^{(n'-n)/2}\,\frac{\mathrm{d}^{n'}}{\mathrm{d}y^{n'}}\,[y^{n}\mathrm{e}^{-y}] \quad (102)$$

where $y = |\alpha|^2$ and $\varphi(\alpha)$ and $\varphi(-\alpha^*)$ are, respectively, the phase angles of α and $-\alpha^*$. To further simplify the expression for $[\hat{D}(\alpha)]_{n,n'}$, it is instructive to establish eq 78 in this representation. Thus we evaluate

$$\sum_{n,\bar{n},n',\bar{n}'}|n,\sigma\rangle\int \mathrm{d}\alpha\ [\hat{D}(\alpha)]_{n,n'}g_{n'}(0)\ g_{\bar{n}'}{}^*(0)[\hat{D}(-\alpha)]_{\bar{n}',\bar{n}}\langle\bar{n},\sigma| \quad (103)$$

It is convenient to carry out the integral over the complex α-plane in polar coordinates. The integral over the phase angle is trivial and leads to

$$\int \mathrm{d}\alpha\ [\hat{D}(\alpha)]_{n,n'}[\hat{D}(-\alpha)]_{\bar{n}',\bar{n}} = \frac{\delta_{n-n',\bar{n}-\bar{n}'}\pi}{\sqrt{n!n'!\bar{n}!\bar{n}'!}}\int_0^{\infty}\mathrm{d}y\ \mathrm{e}^{y}y^{n_+-n_-}\ \frac{\mathrm{d}^{n_+}}{\mathrm{d}y^{n_+}}[y^{n_-}\mathrm{e}^{-y}]\frac{d^{\bar{n}_+}}{\mathrm{d}y^{\bar{n}_+}}[y^{\bar{n}_-}\mathrm{e}^{-y}] \quad (104)$$

where n_+ (n_-) is the greater (lesser) of n and n', and $\bar{n}_+$ and $\bar{n}_-$ are similarly defined. Now $\mathrm{e}^{y}y^{n_+-n_-}\ \mathrm{d}^{n_+}/\mathrm{d}y^{n_+}\ [y^{n_-}\mathrm{e}^{-y}]$ is a polynomial of degree n_+, and hence, if we perform $\bar{n}_+$ parts integrations, we find that the integral vanishes if $\bar{n}_+ > n_+$. Similarly, $\mathrm{e}^{y}y^{\bar{n}_+-\bar{n}_-}\ \mathrm{d}^{\bar{n}_+}/\mathrm{d}y^{\bar{n}_+}\ [y^{\bar{n}_-}\mathrm{e}^{-y}]$ is a polynomial of degree $\bar{n}_+$, and hence, n_+ integrations-by-parts lead us to the conclusion that if $n_+ > \bar{n}_+$, the integral also vanishes. Therefore, the integral is nonzero only if $n_+ = \bar{n}_+$. The term of highest degree of each in the aforementioned polynomials has a coefficient of unity, which allows us to evaluate the integral when $n_+ = \bar{n}_+$ by parts integration. We find that

$$\int d\alpha [\hat{D}(\alpha)]_{n,n'} [\hat{D}(-\alpha)]_{\bar{n}',\bar{n}} = \pi \delta_{n,\bar{n}} \delta_{n',\bar{n}'} \qquad (105)$$

and hence

$$\sum_{n,\bar{n},n',\bar{n}'} |n,\sigma\rangle \int d\alpha\, [\hat{D}(\alpha)]_{n,n'} g_{n'}(0)\, g_{\bar{n}'}{}^*(0) [\hat{D}(-\alpha)]_{\bar{n}',\bar{n}} \langle \bar{n},\sigma| = \pi \sum_n |n,\sigma\rangle\langle n,\sigma| \sum_{n'} g_{n'}(0)\, g_{\bar{n}'}{}^*(0) = \pi \langle g(0)|g(0)\rangle \hat{1} \qquad (106)$$

From the foregoing it is apparent that the polynomial $e^y\, d^{n_+}/dy^{n_+}\, [y^{n_-} e^{-y}]$ of degree n_- belongs to a set of polynomials with fixed $(n_+ - n_-)$ that are mutually orthogonal on the positive real axis under the weight $e^{-y} y^{n_+ - n_-}$. Thus, to within a normalization, they must be the *associated Laguerre polynomials* that are orthogonal on the same interval and under the same weight.[28] Specifically,

$$e^y \frac{d^{n_+}}{dy^{n_+}} [y^{n_-} e^{-y}] = (-1)^{n_+} \frac{n_-!}{n_+!} L_{n_+}^{n_+ - n_-}(y) \qquad (107)$$

Finally, from eq 102 we have that

$$[\hat{D}(\alpha)]_{n,n'} = (-1)^{n_+} \sqrt{\frac{n_-!}{(n_+!)^3}}\, e^{-y/2} y^{(n_+ - n_-)/2} L_{n_+}^{n_+ - n_-}(y) e^{i(n_- - n_+)\varphi} \qquad (108)$$

where φ is the phase angle of α if $n' > n$ and the phase angle of $-\alpha^*$ if $n > n'$. The coherent state with eigenvalue α can be generated from the basic vacuum state, with $\alpha = 0$ by

$$|\phi_0^\sigma(\alpha)\rangle = \hat{D}(\alpha)|\phi_0^\sigma(0)\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}_0^\dagger} |\phi_0^\sigma(0)\rangle \qquad (109)$$

The last equality follows from the first equality in eq 89 and the fact that

$$e^{-\alpha^* \hat{a}_{0,\sigma}} |\phi_0^\sigma(0)\rangle \equiv |\phi_0^\sigma(0)\rangle \qquad (110)$$

The annihilation operator for the α-shifted state is $\hat{a}_{\alpha,\sigma}$ given by eq 91.

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